Noninvertible symmetry and Topological stability of solitons

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The last part is based on arXiv:2210.13780, 2307.00939
with Shi Chen (U. Minnesota)

Contents

- Introduction (Power of symmetry)
- (Quick) review of Noninvertible symmetry in QFT
- Topological stability of soliton beyond homotopy groups (the work with Shi Chen (U. Minnesota), 2210.13780, 2307.00939)

Solving Quantum Field Theories

UV ← microscopic description
- Quantum Chromodynamics
- (Lattice) quantum many-body systems
- Conformal field theories (CFT_{UV})

Renormalization Group (RG) flow

IR ← macroscopic description
- Nonlinear σ-model (chiral Lagrangian)
- Fermi liquid, spin liquid (TQFT), ...
- CFT_{IR}.

This is a very tough problem!

Power of Symmetry

Sometimes, we can know about low-energy dynamics

using Symmetry **without solving microscopic Hamiltonian.**

e.g. In '60s, people didn't know about Quantum Chromodynamics (QCD).

fundamental theory of strong interaction.

current algebra (chiral effective Lagrangian)

$\Longrightarrow$ successful description of low-energy properties of strong interaction

Why this was possible?

Universality due to SSB of chiral symmetry.

Continuous Symmetry in QFT

Noether : If classical action $S[\phi]$ is invariant under continuous transformation,

then there is a conserved current J^μ :

$$\partial_\mu J^\mu = 0.$$

$$\Downarrow$$

In QFT, this becomes Ward-Takahashi identity :

$$\langle \partial_\mu J^\mu(x)\ \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = \sum_i \delta(x-x_i)\ \langle \mathcal{O}_1(x_1)\cdots \delta\mathcal{O}_i(x_i)\cdots\mathcal{O}_n(x_n)\rangle.$$

$$\Downarrow$$

Various theorems related to symmetry.

(such as Nambu-Goldstone theorem, 't Hooft anomaly matching, etc.)

WT identity provides a powerful tool to analyze QFTs.

$\Rightarrow$ We'd like to extend its scope!

Generalization of Symmetry

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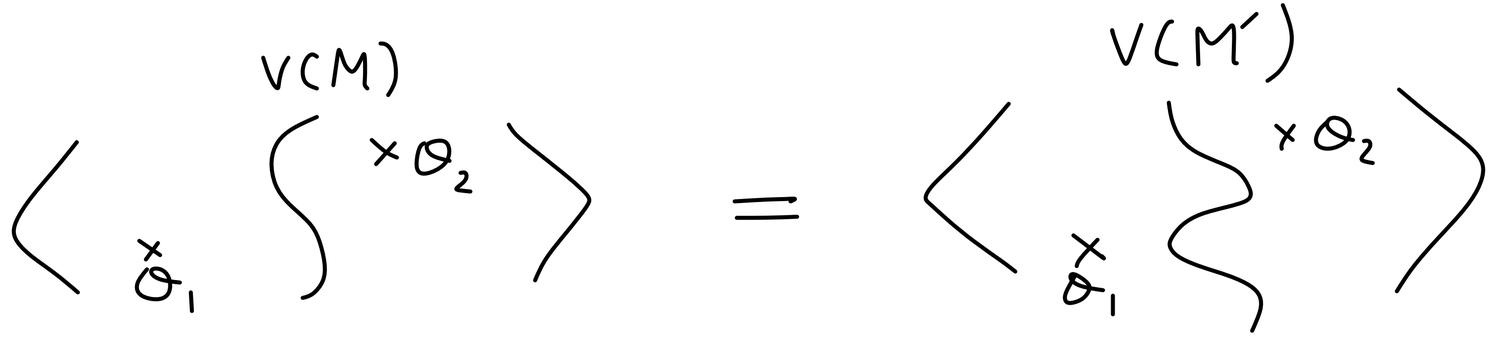
Generalization of Ward-Takahashi identity

Modern Definition of (Generalized) Symmetry

Take-home message

> Symmetry = Topological defect operators

Topological



$\Longleftrightarrow$ Conservation Law

Ordinary Symmetry in Modern Viewpoints

d-dim. QFT has a global symmetry G.

$\overset{\text{def}}{\Longleftrightarrow}$

- $\exists\, V_g(M_{d-1})$: topological codim-1 defect operator for each $g \in G$

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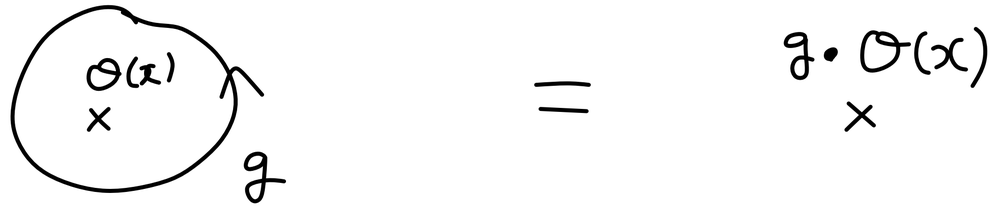
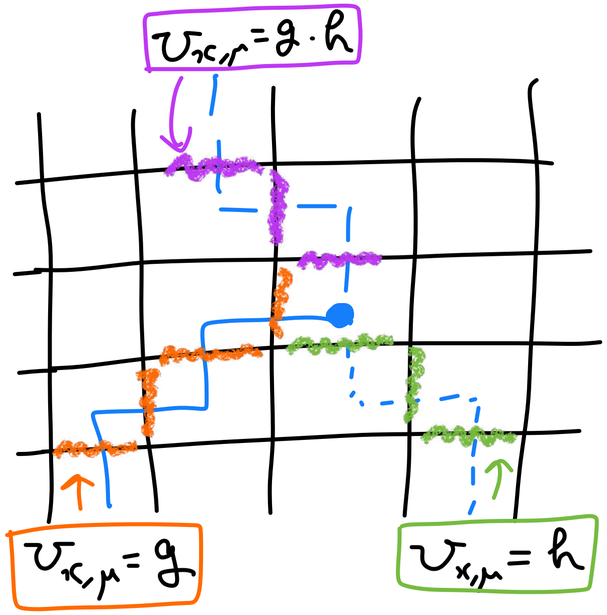
- 

(Valid for both continuous and discrete symmetries)

Network of topological defects

Model $\quad S[\phi] = K \sum_{x,\mu} \left(\phi^\dagger_{x+\mu} \cdot \phi_x + \phi^\dagger_x \cdot \phi_{x+\mu} \right) + V[\phi] \qquad (\phi_x \in \mathbb{C}^N)$

Assume that G is a symmetry (For $g \in G$, $\phi_x \mapsto R(g) \cdot \phi_x$ w/ $R(g) \in U(N)$).



$\Leftrightarrow$

$$S_{\text{twist}} = K \sum_{x,\mu} \left(\phi^\dagger_{x+\mu} R(U_{x,\mu}) \phi_x + \phi^\dagger_x \cdot R(U_{x,\mu})^\dagger \cdot \phi_{x+\mu} \right) + V[\phi]$$

with the twisted hopping (i.e. "link variable")

$U_{x,\mu} \in G$.

(Network of topological defects (consistent with fusion rule))

(Background G gauge field (with flatness condition))

Various generalizations

- **p-form symmetry** (Gaiotto, Kapustin, Seiberg, Willet '14)

  Topological defects have codim-(p+1): $V_g(M_{d-p-1})$.

  (* Esp., 1-form symmetry generalizes the center sym. in gauge theories.)

- **n-group symmetry** (Sharpe '15, Cordova, Dumitrescu, Intriligator '17, YT, Ünsal '19, Hidaka, Nitta, Yokokura '20, ...)

  ≈ Mixture of 0-, 1-, ···, (n-1)-form symmetries.

- **non-invertible symmetry** (Bhardwaj, Tachikawa '17, ... in 2d QFTs.
  '21 ~ higher dim: Nguyen, YT, Ünsal, Koide, Nagoya, Yamaguchi, Choi. Cordova, Hsin, Lam, Shao, Kaidi, Ohmori. Zheng, ...)

  Transformation rule does not form a group

  (We'll discuss more later.)

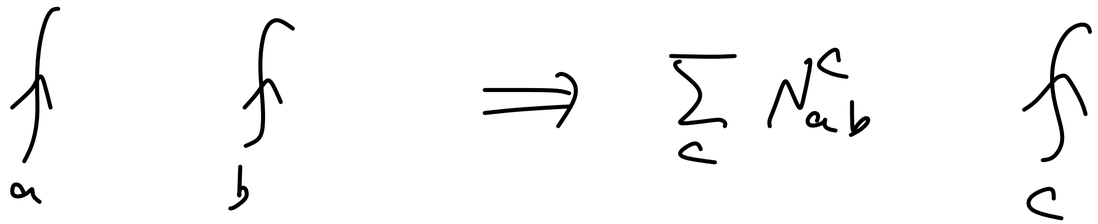
Review of Noninvertible symmetry in QFT

Noninvertible Symmetry

Usually, symmetry forms a group:

Is this necessary?

More general fusion rule: "noninvertible" (or "categorical") symmetry



Why do we consider it?

Massless Majorana in 2d $\xleftrightarrow{\text{JW trans}}$ Critical Ising in 2d

Local op : $1, \psi_L, \psi_R, \varepsilon$ $\qquad$ $1, \sigma, \varepsilon$ (spin, energy)

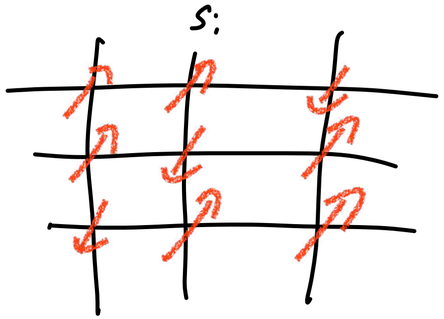
Sym : $(\mathbb{Z}_2)_F \times (\mathbb{Z}_2)_L$ $\qquad$ $(\mathbb{Z}_2)_{\text{spin}}$

$\psi_{L,R} \to -\psi_{L,R}$ $\qquad$ $\psi_L \to -\psi_L$, $\psi_R \to \psi_R$ $\qquad$ $\sigma \to -\sigma$

Dual symmetry

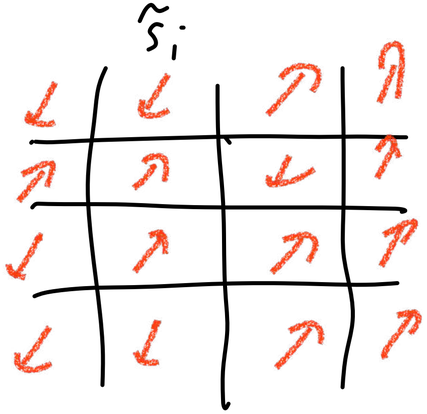
$\rightsquigarrow$ What is the counterpart of $(\mathbb{Z}_2)_L$ in Ising side?

Kramers - Wannier duality of 2d Ising model



$e^{J \sum s_i s_j}$

Dual



$e^{\tilde{J} \sum \tilde{s}_i \tilde{s}_j}$

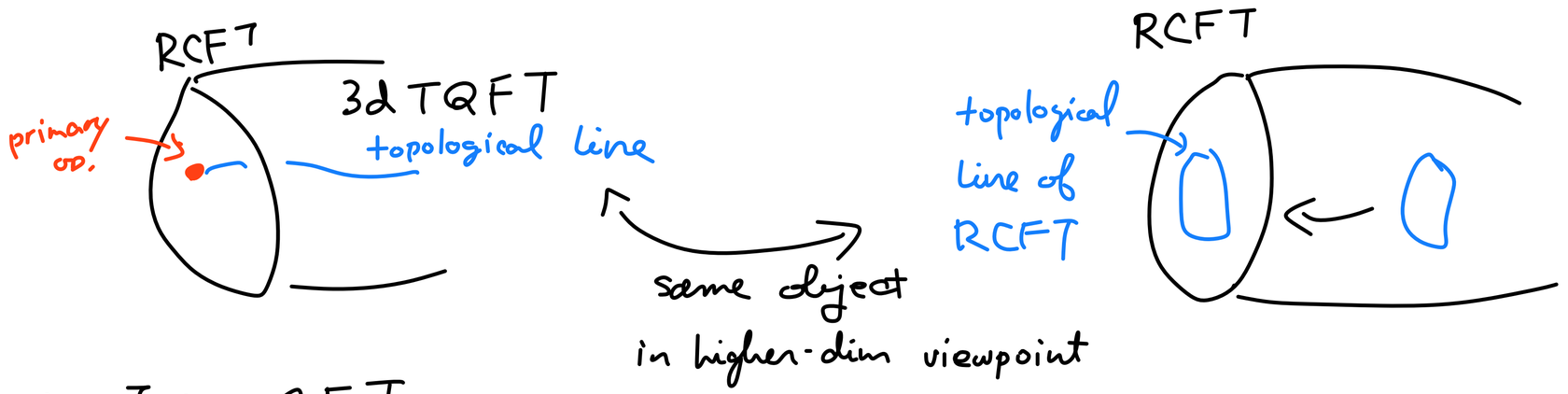
This exchanges the original spin & dual spin.

$$s(x) \longleftrightarrow \tilde{s}(x).$$

Not mutually local

What's this trans?

A. Non-invertible symmetry! (← described by fusion category)



(cf. Verlinde '88)

2d Ising CFT

Primaries : $\underset{\text{trivial}}{1}$, $\underset{\text{energy}}{\varepsilon}$, $\underset{\text{spin}}{\sigma}$

Operator Product Expansion

$$\begin{cases} \sigma \times \sigma = 1 + \varepsilon \\ \sigma \times \varepsilon = \sigma \\ \varepsilon \times \varepsilon = 1 \end{cases}$$

Topological line : 1 , $\underset{\mathbb{Z}_2 \text{ sym}}{\eta}$, $\underset{\text{KW duality}}{D}$

$$\begin{cases} D \times D = 1 + \eta \\ D \times \eta = D \\ \eta \times \eta = 1 \end{cases}$$

(Lattice construction : Aasen, Fendley, Mong $\overset{\text{extension to 4d}}{\Longrightarrow}$ Koide, Nagoya, Yamaguchi; Choi, Cordova, et al. ; Kaidi, Ohmori, Zheng ...)

Non-invertible 1-form symmetry

(Nguyen, YT, Ünsal ; Heidenreich, McNamara, Montero, Reece, Rudelius, Valenzuela, ...)

Pure $U(1)$ gauge theory has $U(1)^{[1]}$ and $(\mathbb{Z}_2)_C$ (charge conjugation).

Gauging $(\mathbb{Z}_2)_C$ $\Longrightarrow$ Gauge group becomes $O(2) = U(1) \rtimes \mathbb{Z}_2$.

↑ Center of $O(2)$ is just $\mathbb{Z}_2$, which is much smaller than $U(1)$.

Do we lose most part of 1-form symmetry by gauging $(\mathbb{Z}_2)_C$?

A. They survive as non-invertible symmetry!

$U_\alpha(M_{d-2})$: $U(1)$ 1-form symmetry generator in $U(1)$ gauge theory

$\Downarrow$ $(\mathbb{Z}_2)_C$ gauging

$U_\pi(M_{d-2})$: $\mathbb{Z}_2^{[1]}$ defect for $O(2)$ gauge theory

$U'_\alpha(M_{d-2}) = U_\alpha + U_{-\alpha}$: Non-invertible 1-form symmetry generators of $O(2)$ gauge theory.

$$\left(U'_\alpha U'_\beta = U'_{\alpha+\beta} + U'_{\alpha-\beta} \right)$$

[In 3d $O(2)$ gauge theory with monopoles, selection rule of confining strings obey this non-inv. symmetry.]

Application : Solitonic symmetry beyond homotopy

Based on my work with Shi Chen

When (usual) symmetry is spontaneously broken

$$G \xrightarrow{SSB} H,$$

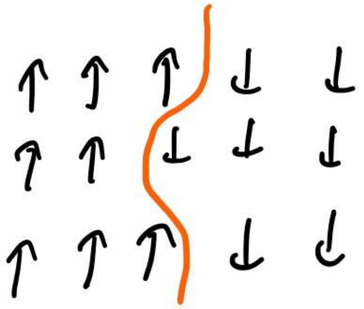
the low-energy effective theory is given by a non-linear σ-model:

$$\sigma : \text{Spacetime} \longrightarrow G/H$$

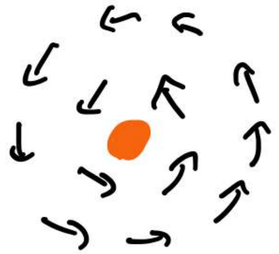
$$Z = \int \mathcal{D}\sigma \exp\left(-\frac{1}{g^2}\int |d\sigma|^2 + (\cdots)\right).$$

Small fluctuations of σ : Nambu-Goldstone modes

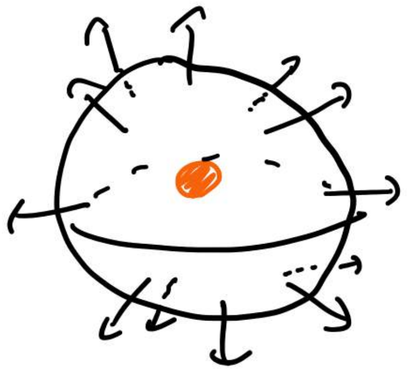
Large fluctuation : Topological Solitons



domain wall



vortex

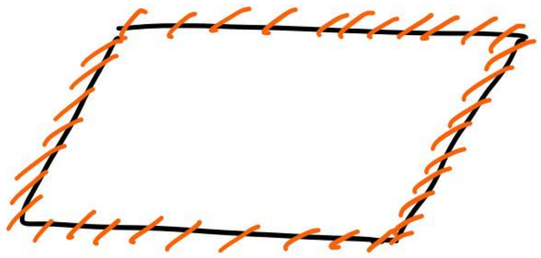


hedgehog

Topological Stability and Homotopies

(Mermin '79 Rev. Mod. Phys.)

Spacetime



$\xrightarrow{\sigma}$ $\mathcal{M} (= G/H)$.

To find finite action/energy (density) $\int |d\sigma|^2 < \infty$, its convenient to identifiy ∞'s of $\mathbb{R}^n$:

$$\mathbb{R}^n \cup \{\infty\} \simeq S^n \xrightarrow{\sigma} \mathcal{M}.$$

$\Rightarrow$ Topoligcal solitons are classified by homotopies of the target space

$$\pi_n(\mathcal{M}).$$

Conventional Wisdom: Solitonic Sym. $\cong$ Hom $(\pi_n(\mathcal{M}), U(1))$.

↳ Is this always true?

Claim There is a more rich structure for solitonic symmetries.

4d $\mathbb{C}P^1$ σ-model

There are two kinds of solitons:

$\pi_2(\mathbb{C}P^1) \simeq \mathbb{Z}$ $\longleftrightarrow$ stable vortex

$\pi_3(\mathbb{C}P^1) \simeq \mathbb{Z}$ $\longleftrightarrow$ stable particle (Hopfion)

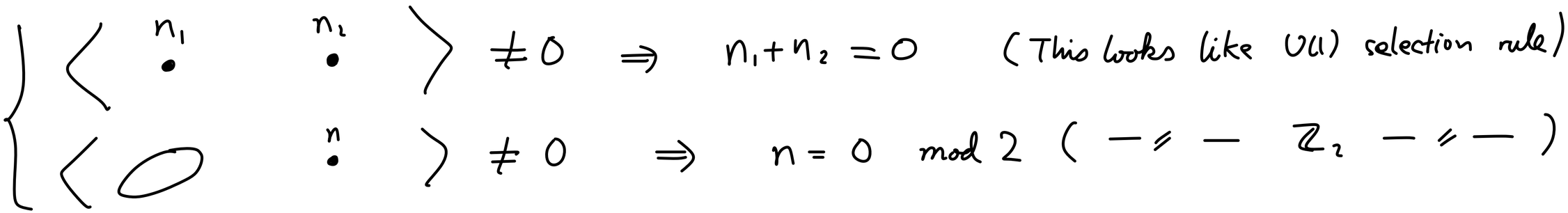
Hopf fibration $S^1 \to S^3 \to S^2$

This homotopy argument has been understood as the presence of solitonic U(1) symmetries.

$\pi_2(\mathbb{C}P^1) \simeq \mathbb{Z}$ $\longleftrightarrow$ U(1) 1-form symmetry

$\pi_3(\mathbb{C}P^1) \simeq \mathbb{Z}$ $\longleftrightarrow$ ~~U(1) 0-form~~ U(1) symmetry

Non-invertible symmetry $\supset \mathbb{Z}_2$ subgroup



(↑ This follows from the result of Pontryagin '41)

Topological operators

ω_{S^2} ← volume 2-form of unit S^2 $=: \frac{da}{2\pi}$ $\quad (a : U(1)$ gauge field $)$

$\mathbb{Z}_2$ symmetry generator : ← $\pi \times$ Hopf inv.

$$\mathcal{H}_\pi(M_3) = \exp\left(i \int_{M_3} \frac{a\,da}{4\pi}\right)$$

$\uparrow$

$\widetilde{\Omega}^{\text{Spin}}_3(\mathbb{CP}^1) \simeq \mathbb{Z}_2$

This generates "π rotation" for Hopf charge.

For a fractional rotation $\alpha = \frac{\pi}{N}$, we can construct

$$\mathcal{H}_{\frac{\pi}{N}}(M_3) = \int \mathcal{D}b \exp\left(i \frac{N}{4\pi} \int_{M_3} b \cup db + i \frac{1}{2\pi} \int_{M_3} b \cup da\right).$$

↳ Non-invertible Hopf number symmetry.

(Generator is a partition function of 3d topological QFTs)

Summary

Take-home message

> Symmetry = Topological defect operators

$\Rightarrow$ New aspects of strongly-coupled QFTs.